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Vibrations of double-string complex system under moving forces. Closed solutions

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A B S T R A C T

In this paper the dynamic response of a double-string system traversed by a constant or a harmonically oscillating moving force is considered. The force is moving with a constant velocity on the top string. The strings are identical, parallel, one upon the other and continuously coupled by a linear Winkler elastic element. The classical solution of the response of a double-string system subjected to a force moving with a constant velocity has a form of an infinite series. The main goal of this paper is to show that in the considered case a part of the solution can be presented in a closed, analytical form instead of an infinite series. The presented method of finding the solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations.

1. Introduction

The problem of a dynamic response of a structure subjected to moving loads is interesting and important. This problem occurs in dynamics of bridges, roadways, railways and runways as well as missiles, aircrafts and other structures. Various types of structures and girders like beams, plates, shells, frames have been considered. Also various models of moving loads have been assumed [1]. Both deterministic and stochastic approaches have been presented [2–4]. A string as a simple model of a one-dimensional continuous system resistant to tension but not to bending is often used in analysis of numerous engineering structures and has been a subject of great scientific interest for a considerable time. This follows from the fact that the vibrations of a string are described by the wave differential equation. This allows one to see the wave effect in a string, contrary to many more complex systems where it might be either not present or not clearly visible. Various aspects of the dynamics response of a string under a moving load have been considered, among others, in the papers [5–8]. An important technological extension of a single string, beam or plate is that of the double-string, double-beam or string-beam system. Various aspects of the dynamic response of a double-string and double-beam system have been considered by Oniszczuk [9–13]. In his papers an excellent bibliography is also given. Free and forced vibrations of a double-beam system have been considered, among others, in the papers [14,15]. The problem of vibration and buckling of a double-beam system under compressive axial loading is presented in the paper [16]. Vibrations of a complex system under moving load have been studied in many papers [17,18].

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In this paper the dynamic behavior of a double-string system traversed by a constant or a harmonically oscillating moving force is considered. The force is moving with a constant velocity on the top string. The strings are identical, parallel one upon the other and continuously coupled by a linear Winkler elastic element.

The classical solution of the response of a double-string system subjected to a force moving with a constant velocity has a form of an infinite series. The main goal of this paper is to show that in the considered case a part of the solution can be presented in a closed, analytical form instead of an infinite series. Using the method of superposed deflections Kączkowski [19] has shown for a simply supported Euler–Bernoulli beam that the aperiodic part of the solution can be presented in a closed form. Next, Reipert obtained a closed form solution for a beam with arbitrary boundary conditions [20] and for a frame [21]. In this paper we use a different method to obtain the solutions in a closed form. The presented method of finding the solution in a closed form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. This method has been used to find closed, analytical solution for a finite, simply supported Timoshenko beam loaded by point force moving with a constant velocity [22].

2. Forced vibrations. General solution

Let us consider the vibrations of a connected double equal string complex continuous system excited by a point force $P(t)$ moving with a constant velocity v as shown in Fig. 1. The differential equations of motion of the springs system have the form

$$-S\frac{\partial^2 w_1(x,t)}{\partial x^2}+k[w_1(x,t)-w_2(x,t)]+m\frac{\partial^2 w_1(x,t)}{\partial t^2}=P(t)\delta(x-vt),$$
$$-S\frac{\partial^2 w_2(x,t)}{\partial x^2}+k[w_2(x,t)-w_1(x,t)]+m\frac{\partial^2 w_2(x,t)}{\partial t^2}=0, \tag{1}$$

where S is the tension of the string, m is the mass of the string, k is the stiffness modulus of a Winkler elastic element and $\delta(\cdot)$ is the Dirac delta.

The boundary conditions have the form

$$w_1(0,t)=0, \quad w_1(L,t)=0, \quad w_2(x,0)=0, \quad w_2(L,t)=0. \tag{2}$$

After introducing the dimensionless variables

$$\xi=\frac{x}{L}, \quad T=\frac{vt}{L}, \quad \xi\in[0,1], \quad T\in[0,1] \tag{3}$$

Eq. (1) takes the form

$$-w_1^{II}(\xi,T)+k_0[w_1(\xi,T)-w_2(\xi,T)]+\eta^2\ddot{w}_1(\xi,T)=P_0(T)\delta(\xi-T), \quad -w_2^{II}(\xi,T)+k_0[w_2(\xi,T)-w_1(\xi,T)]+\eta^2\ddot{w}_2(\xi,T)=0, \tag{4}$$

where

$$v_s=\sqrt{\frac{S}{m}}, \quad \eta=\frac{v}{v_s}, \quad k_0=\frac{kL^2}{S}, \quad P_0(T)=P(t)\frac{L}{S}.$$

The quantity v_s represents the wave velocity in the string. The Roman numerals denote differentiation with respect to the spatial coordinate ξ, and the dots denote differentiation with respect to time T.

The boundary conditions (2) have the form

$$w_1(0,T)=w_1(1,T)=0, \quad w_2(0,T)=w_2(1,T)=0. \tag{5}$$

Let the initial conditions have the form

$$w_1(\xi,0)=0, \quad \dot{w}_1(\xi,0)=0, \quad w_2(\xi,0)=0, \quad \dot{w}_2(\xi,0)=0. \tag{6}$$

Let us introduce two new functions

$$w_I(\xi,T)=w_1(\xi,T)+w_2(\xi,T), \tag{7}$$

and

$$w_{II}(\xi,T)=w_1(\xi,T)-w_2(\xi,T). \tag{8}$$

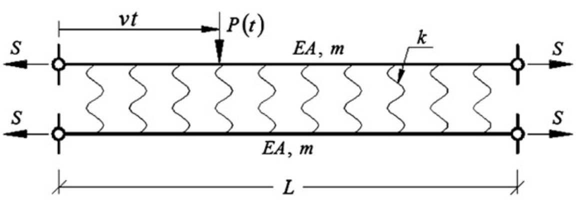

Fig. 1. Double-string system under moving force.

From Eq. (4) we obtain two new differential equations for the functions $w_I(\xi,T)$ and $w_{II}(\xi,T)$

$$-w_I^{II}(\xi,T)+\eta^2\ddot{w}_I(\xi,T)=P_0(T)\delta(\xi-T), \tag{9}$$

and

$$-w_{II}^{II}(\xi,T)+\eta^2\ddot{w}_{II}(\xi,T)+2k_0w_{II}(\xi,T)=P_0(T)\delta(\xi-T). \tag{10}$$

Eq. (9) describes vibrations of a single string, while Eq. (10) describes vibrations of a single string resting on an elastic, Winkler support with parameter $2k$, $(2k_0)$. From (7) and (8) it follows that

$$w_1(\xi,T)=\frac{w_I(\xi,T)+w_{II}(\xi,T)}{2},\quad w_2(\xi,T)=\frac{w_I(\xi,T)-w_{II}(\xi,T)}{2}. \tag{11}$$

The solutions to Eqs. (9) and (10) for boundary conditions (5) are assumed to be in the form of the sine series

$$w_J(\xi,T)=\sum_{n=1}^{\infty}y_{Jn}(T)\sin n\pi\xi, \tag{12}$$

where $J=I$ or $J=II$.

By substituting expression (12) into Eqs. (9) and (10) and using the orthogonalization method one obtains the following set of uncoupled ordinary differential equations:

$$\ddot{y}_{Jn}(T)+\overline{\omega}_{Jn}^2y_{Jn}(T)=\frac{2P_o(T)}{\eta^2}\sin n\pi T, \tag{13}$$

where for $J=I$ $\overline{\omega}_{Jn}=\overline{\omega}_{In}=n\pi/\eta$, and for $J=II$ $\overline{\omega}_{Jn}=\overline{\omega}_{IIn}=\sqrt{(n\pi)^2+2k_0}/\eta$.

These functions fulfill the initial conditions

$$y_{In}(0)=0,\quad \dot{y}_{In}(0)=0,\quad y_{IIn}(0)=0,\quad \dot{y}_{IIn}(0)=0. \tag{14}$$

The solution to Eq. (13) has the form

$$y_{Jn}(T)=\frac{2}{\eta^2\overline{\omega}_{Jn}}\int_0^T\sin\overline{\omega}_{Jn}(T-\tau)\sin n\pi\tau P_0(\tau)d\tau,\quad \text{for } 0\le T\le 1, \tag{15}$$

and

$$y_{Jn}(T)=\frac{2}{\eta^2\overline{\omega}_{Jn}}\int_0^1\sin\overline{\omega}_{Jn}(T-\tau)\sin n\pi\tau P_0(\tau)d\tau,\quad \text{for } T\ge 1. \tag{16}$$

After integrating Eq. (15) we obtain the classical solution for the vibrations of the system of the strings in the form of infinite series. Below we present a method for finding the solution also in a closed analytic form.

Let the function $f(\xi,T)$ for $\xi\in[0,1]$, $T\in[0,1]$ be given by the series

$$f(\xi,T)=\sum_{n=1}^{\infty}\frac{b_1(n\pi)^{2r}\sin n\pi T\sin n\pi\xi+b_2(n\pi)^{2s-1}\cos n\pi T\sin n\pi\xi}{a_0(n\pi)^{2k}+a_1(n\pi)^{2(k-1)}+\cdots+a_{k-1}(n\pi)^2+a_k}, \tag{17}$$

where $k,r,s\in\mathbf{N}_0$; quantities a_i (i=0,2, …, k) are real numbers such that $a_0\neq 0$. To function (17) one can associate the following partial differential equation:

$$a_0(-1)^k\frac{d^{2k}f(\xi,T)}{d\xi^{2k}}+a_1(-1)^{k-1}\frac{d^{2(k-1)}f(\xi,T)}{d\xi^{2(k-1)}}+\cdots+a_{k-1}(-1)\frac{d^2f(\xi,T)}{d\xi^2}+a_kf(\xi,T)$$
$$=b_1(-1)^r\frac{d^{2r}\delta(\xi-T)}{d\xi^{2r}}+b_2(-1)^{s-1}\frac{d^{2s-1}\delta(\xi-T)}{d\xi^{2s-1}}, \tag{18}$$

for which it is a solution for the boundary conditions

$$[f(0,T)]^{(2j)}=[f(1,T)]^{(2j)}=0, \tag{19}$$

where $[f(\xi,T)]^{(2j)}=d^{2j}f(\xi,T)/d\xi^{2j}$ and j=0,1, …, $k-1$.

This can be verified by solving (18) using finite Fourier sine transform. After solving (18) by, for example, Laplace transform and taking into account the boundary conditions (19) we get the function $f(\xi,T)$ in a closed form.

While solving (18) one has to take into account the relationship

$$\int_0^1\frac{d^{2r}\delta(\xi-T)}{d\xi^{2r}}\sin n\pi\xi d\xi=(-1)^r(n\pi)^{2r}\sin n\pi T, \tag{20}$$

$$\int_0^1\frac{d^{2s-1}\delta(\xi-T)}{d\xi^{2s-1}}\sin n\pi\xi d\xi=(-1)^{s-1}(n\pi)^{2s-1}\cos n\pi T. \tag{21}$$

Now we consider two particular cases of the load processes, one if the moving point force is constant and second if the moving force is harmonically oscillating. Besides the classical solutions in a form of an infinite series also the closed, analytical solutions have been obtained using the method presented above.

3. Moving constant force

Let the moving force be constant, $P_0(T)=P_0=PL/S=const$. From Eq. (16) it follows that the solutions to Eqs. (9) and (10) for the initial condition (6) are sums of particular integrals $w_I^A(\xi,T)$, $w_{II}^A(\xi,T)$ and general integrals $w_I^S(\xi,T)$, $w_{II}^S(\xi,T)$:

$$w_I(\xi,T)=w_I^A(\xi,T)+w_I^S(\xi,T)=\frac{2P_o}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\sin n\pi T\sin n\pi\xi}{(n\pi)^2}-\frac{2P_o\eta}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{\eta}T\sin n\pi\xi}{(n\pi)^2}, \tag{22}$$

and

$$w_{II}(\xi,T)=w_{II}^A(\xi,T)+w_{II}^S(\xi,T)=2P_o\sum_{n=1}^{\infty}\frac{\sin n\pi T\sin n\pi\xi}{(n\pi)^2(1-\eta^2)+2k_o}-2P_o\,\eta\sum_{n=1}^{\infty}\frac{n\pi\sin\left[\eta^{-1}\sqrt{(n\pi)^2+2k_oT}\right]\sin n\pi\xi}{\sqrt{(n\pi)^2+2k_o}[(n\pi)^2(1-\eta^2)+2k_o]}. \tag{23}$$

The dynamic component of the tension in the strings is given by

$$N_i(\xi,T)=EA\frac{\partial w_i(x,t)}{\partial x}=\frac{EA}{L}\frac{\partial w_i(\xi,T)}{\partial \xi}. \tag{24}$$

The differentiation of Eqs. (22) and (23) in order to obtain the tension (24) results with slowly convergent series. These remarks refer to the next solutions in this paper as well.

The functions $w_I^A(\xi,T)$ and $w_{II}^A(\xi,T)$ are aperiodic vibrations and $w_I^S(\xi,T)$ and $w_{II}^S(\xi,T)$ are free vibrations of the strings. Now we will present the aperiodic solutions $w_I^A(\xi,T)$ given by the first series in expression (22) in a closed analytical form using the method described by Eqs. (17)–(21).

Let us notice an important fact that this function is a solution not only of the partial differential equations (9) but also of the ordinary equation

$$-[w_I^A(\xi,T)]^{II}=\frac{P_o}{1-\eta^2}\delta(\xi-T), \tag{25}$$

for the boundary conditions (5).

The variable T in Eq. (25) is the only parameter that describes the location of the moving force on the string. After solving Eq. (25) we can obtain the functions $w_I^A(\xi,T)$ in a closed form instead of a series. The closed form of the solution has the form

$$w_I^A(\xi,T)=\begin{cases}\dfrac{P_o}{1-\eta^2}(1-T)\xi & \text{for } \xi\le T,\\ \dfrac{P_o}{1-\eta^2}T(1-\xi) & \text{for } \xi\ge T,\end{cases} \tag{26}$$

or in the short form

$$w_I^A(\xi,T)=\frac{P_o}{1-\eta^2}\left[(1-T)\xi-(\xi-T)H(\xi-T)\right], \tag{27}$$

where $H(\cdot)$ is the unit step Heaviside function.

Also the function $w_I^S(\xi,T)$ can be presented in the closed form. For the interval $i\eta\le T\le(i+1)\eta\le 1$, where $i=0,2,4,\ldots,n$, the function $w_I^S(\xi,T)$is also a solution of the equation

$$[w_I^S(\xi,T)]^{II}=\frac{P_o\eta}{1-\eta^2}\delta\left[\xi-\left(\frac{T}{\eta}-i\right)\right], \tag{28}$$

and hence can be presented in the closed form

$$w_I^S(\xi,T)=-\frac{P_o}{1-\eta^2}\left\{\xi\left[(i+1)\eta-T\right]+\left[\eta(\xi+i)-T\right]H(\xi+i-(T/\eta))\right\}. \tag{29}$$

For the interval $i\,\eta\le T\le(i+1)\,\eta\le 1$ where $i=1,3,5,\ldots,n+1$, the function $w_I^S(\xi,T)$is also a solution of the equation

$$[w_I^S(\xi,T)]^{II}=-\frac{P_o\eta}{1-\eta^2}\delta\left[\xi-\left(i+1-\frac{T}{\eta}\right)\right], \tag{30}$$

and hence can be presented in the closed form

$$w_I^S(\xi,T)=\frac{P_o}{1-\eta^2}\left\{\xi(T-i\eta)-\left[\eta\xi-\eta(i+1)+T\right]H\left[\xi-(i+1-(T/\eta))\right]\right\}. \tag{31}$$

Taking into account the relationships (27), (29) and (31) the function $w_I(\xi,T)=w_I^A(\xi,T)+w_I^S(\xi,T)$ can be presented in a closed analytical form. For the interval $i\eta\le T\le(i+1)\eta\le 1$, where i=0,2, …, n and $\eta<1$ ($v<v_s$) one obtains

$$w_I(\xi,T)=\frac{P_o}{1-\eta^2}\left\{(1-T)\xi-\left[(i+1)\eta-T\right]\xi-(\xi-T)H(\xi-T)\quad+\left[\eta(\xi+i)-T\right]H(\xi+i-(T/\eta))\right\}, \tag{32}$$

and for the interval $i\eta \le T \le (i+1)\eta \le 1$, where i=1,3, …, n+1and $\eta < 1$ one obtains

$$w_I(\xi,T) = \frac{P_o}{1-\eta^2}\{(1-T)\xi + (T-i\eta)\xi - (\xi - T)H(\xi - T) - [\eta\xi - \eta(i+1)+T]H[\xi-(i+1-(T/\eta))]\}. \tag{33}$$

Eq. (32) gives the string response when the front of the free wave moves in the same direction as the point force while Eq. (33) is when it moves in the opposite direction. For example, for $0 \le T \le \eta < 1$ it follows from Eq. (32) (i=0) that

$$w_I(\xi,T) = \begin{cases} \frac{P_o}{1+\eta}\xi & \text{for} \quad \xi \le T, \\ \frac{P_o}{1-\eta^2}(T-\eta\xi) & \text{for} \quad T \le \xi \le \frac{T}{\eta}, \\ 0 & \text{for} \quad \frac{T}{\eta} \le \xi \le 1. \end{cases} \tag{34}$$

From (26) or (27) and (29) for i=0 we obtain the solution for $\eta > 1$, i.e. when the point force velocity is bigger than the transverse wave velocity, namely

$$w_I(\xi,T) = \begin{cases} \frac{P_o}{1+\eta}\xi & \text{for} \quad \xi \le \frac{T}{\eta}, \\ \frac{P_o}{1-\eta^2}(\xi - T) & \text{for} \quad \frac{T}{\eta} \le \xi \le T, \\ 0 & \text{for} \quad \xi \ge T. \end{cases} \tag{35}$$

In the particular case when η=1 (v=v_s), Eqs. (34) and (35) imply that

$$w_I(\xi,T) = \begin{cases} \frac{P_o}{2}\xi & \text{for} \quad \xi < T, \\ 0 & \text{for} \quad \xi > T. \end{cases} \tag{36}$$

Solutions (34), (35) and (36) are presented in Fig. 2.

The aperiodic solution $w_{II}^A(\xi,T)$ given by the first series in the expression (23) is a solution not only of the partial differential equation (10) but also of the ordinary equation

$$-(1-\eta^2)w_{II,\xi\xi}^A(\xi,T) + 2k_o w_{II}^A(\xi,T) = P_o\delta(\xi - T). \tag{37}$$

After solving Eq. (37) one can obtain the functions $w_{II}^A(\xi,T)$ in a closed, analytical form instead of a series. The closed form of the solution has the form for $v < v_s$, ($\eta < 1$)

$$w_{II}^A(\xi,T) = \frac{P_o}{a(1-\eta^2)}\left[\frac{\sinh a(1-T)\sinh a\xi}{\sinh a} - \sinh a(\xi - T)H(\xi - T)\right], \tag{38}$$

where a^2=$2k_0/(1-\eta^2)$ and for $v < v_s$ ($\eta < 1$),

$$w_{II}^A(\xi,T) = \frac{P_o}{b(1-\eta^2)}\left[\frac{\sin b(1-T)\sin b\xi}{\sin b} - \sin b(\xi - T)H(\xi - T)\right], \tag{39}$$

where b^2=$2k_0/(\eta^2-1)$.

In the particular case if v=v_s (η=1) the solution has the form

$$w_{II}^A(\xi,T) = \frac{P_o}{2k_o}\delta(\xi - T). \tag{40}$$

The solution (40) has a Dirac delta singularity. For the general integral $w_{II}^S(\xi,T)$ one cannot find a closed solution by the method presented above.

In the case if $T \ge 1$ (free vibration of the system) from Eq. (16) one obtains

$$w_J(\xi,T) = \frac{2P_0}{\eta^2}\sum_{n=1}^{\infty}\frac{(-1)^n n\pi \sin\overline{\omega}_{Jn}(T-1)\sin n\pi\xi}{\overline{\omega}_{Jn}[\overline{\omega}_{Jn}^2 - (n\pi)^2]} - \frac{2P_0}{\eta^2}\sum_{n=1}^{\infty}\frac{n\pi \sin n\pi T \sin n\pi\xi}{\overline{\omega}_{Jn}[\overline{\omega}_{Jn}^2 - (n\pi)^2]}, \tag{41}$$

where J=I,II.

For example for J=I the solution (41) has the form

$$w_I(\xi,T) = \frac{2P_0\eta}{1-\eta^2}\sum_{n=1}^{\infty}\frac{(-1)^n \sin(n\pi/\eta)(T-1)\sin n\pi\xi}{(n\pi)^2} - \frac{2P_0\eta}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\sin n\pi T \sin n\pi\xi}{(n\pi)^2}. \tag{42}$$

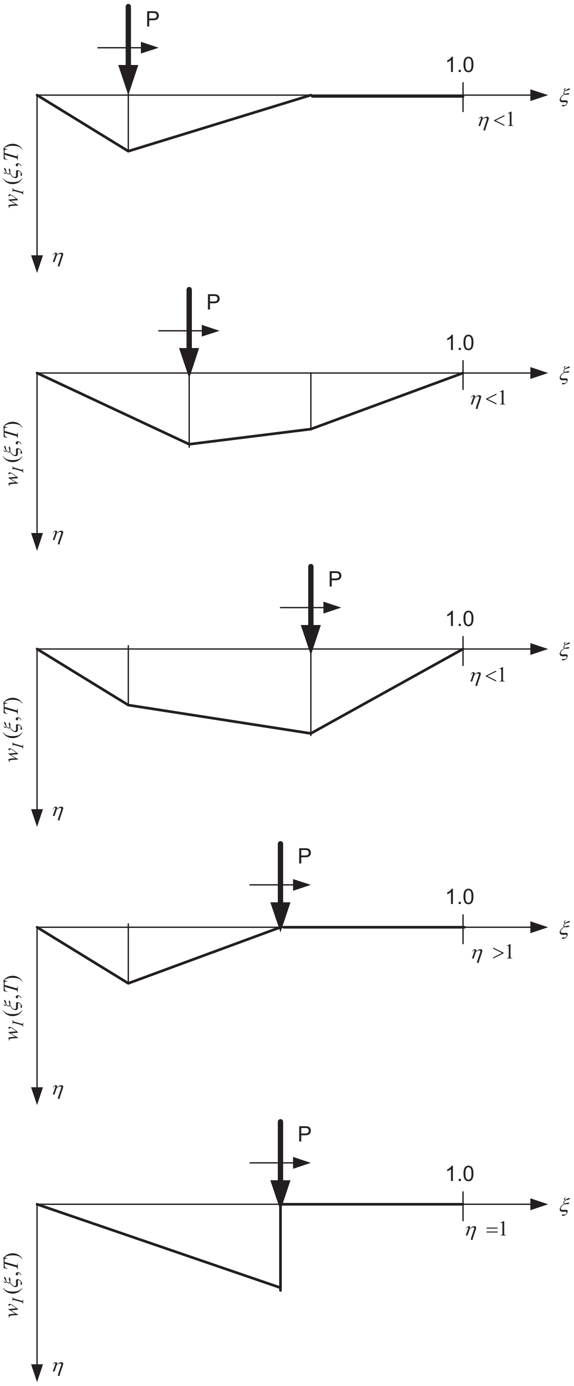

Fig. 2. Transverse displacement of the string under moving force.

4. Moving harmonic oscillation force

Let the moving force be a harmonic oscillator of the form

$$P(t) = P\sin(\omega t + \varphi), \tag{43}$$

hence

$$P_0(T) = P_0\sin(\tilde{\omega} T + \varphi), \tag{44}$$

where $\tilde{\omega} = \omega(L/v)$.

Taking into account (44) and Eq. (15) the response function $w_I(\xi,T)$ has the form

$$w_I(\xi,T) = w_I^A(\xi,T) + w_I^S(\xi,T), \tag{45}$$

where

$$w_I^A(\xi,T) = 2P_o\sin(\tilde{\omega}T+\varphi)\sum_{n=1}^{\infty}\frac{[(n\pi)^2(1-\eta^2)-\eta^2\tilde{\omega}^2]\sin n\pi T\cdot\sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2-2(n\pi)^2\eta^2\tilde{\omega}^2(1+\eta^2)+\eta^4\tilde{\omega}^4}$$
$$-4P_o\eta^2\overline{\omega}\cos(\tilde{\omega}T+\varphi)\sum_{n=1}^{\infty}\frac{n\pi\cos n\pi T\cdot\sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2-2(n\pi)^2\eta^2\tilde{\omega}^2(1+\eta^2)+\eta^4\tilde{\omega}^4}, \tag{46}$$

and

$$w_I^S(\xi,T) =$$
$$2P_o\eta\sum_{n=1}^{\infty}\frac{[(n\pi)^2(1-\eta^2)-\eta^2\tilde{\omega}^2]\sin\left[(n\pi/\eta)T\right]\sin\varphi+2n\pi\tilde{\omega}\eta\cos\left[(n\pi/\eta)T\right]\cos\varphi}{(n\pi)^4(1-\eta^2)^2-2(n\pi)^2\eta^2\tilde{\omega}^2(1+\eta^2)+\eta^4\tilde{\omega}^4}\sin n\pi\xi. \tag{47}$$

Taking into account (44) and Eq. (15) shows that the next response function is

$$w_{II}(\xi,T) = w_{II}^A(\xi,T) + w_{II}^S(\xi,T), \tag{48}$$

where

$$w_{II}^A(\xi,T) = \sum_{n=1}^{\infty}\frac{2P_o\sin(\tilde{\omega}T+\varphi)[(n\pi)^2(1-\eta^2)+k_o-\eta^2\tilde{\omega}^2]\sin n\pi T\sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2+2(n\pi)^2\left[(1-\eta^2)k_o-\eta^2\tilde{\omega}^2(1+\eta^2)\right]+(k_o-\eta^2\tilde{\omega}^2)^2}$$
$$-\sum_{n=1}^{\infty}\frac{4P_o\eta^2\tilde{\omega}\cos(\tilde{\omega}T+\varphi)n\pi\cos n\pi T\sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2+2(n\pi)^2\left[(1-\eta^2)k_o-\eta^2\tilde{\omega}^2(1+\eta^2)\right]+(k_o-\eta^2\tilde{\omega}^2)^2}, \tag{49}$$

and

$$w_{II}^S(\xi,T) = \frac{2P_o\overline{\omega}}{\eta^2}\sum_{n=1}^{\infty}\frac{n\pi}{\omega_{IIn}}\frac{\eta^2\overline{\omega}[\omega_{IIn}^2-(n\pi)^2+\overline{\omega}^2]\sin[\omega_{IIn}T]\sin\varphi-2\omega_{IIn}\cos[\omega_{IIn}T]\cos\varphi}{\left[\omega_{IIn}^2-(n\pi)^2-\overline{\omega}^2\right]^2-4(n\pi)^2\overline{\omega}^2}\sin n\pi\xi, \tag{50}$$

and

$$\omega_{IIn}^2 = \frac{(n\pi)^2+k_o}{\eta^2}.$$

The aperiodic vibration $w_I^A(\xi,T)$ given by Eq. (46) is also a solution of the ordinary differential equation

$$(1-\eta^2)^2[w_I^A(\xi,T)]^{IV}+2\eta^2\tilde{\omega}^2(1+\eta^2)[w_I^A(\xi,T)]^{II}+\eta^4\tilde{\omega}^4 w_I^A(\xi,T)$$
$$=-P_o\sin(\tilde{\omega}T+\varphi)[(1-\eta^2)\delta^{II}(\xi-T)+\eta^2\tilde{\omega}^2\delta(\xi-T)]+2P_o\eta^2\tilde{\omega}\cos(\tilde{\omega}T+\varphi)\delta^I(\xi-T), \tag{51}$$

and satisfies the boundary conditions

$$w_I^A(0,T) = w_I^A(1,T) = 0, \quad [w_I^A(0,T)]^{II} = [w_I^A(1,T)]^{II} = 0. \tag{52}$$

After taking into account the boundary conditions (52), the solution to Eq. (51) for the case $\eta \neq 1$ takes the form

$$w_I^A(\xi,T) = \frac{P_o}{2\eta\tilde{\omega}}\left(\frac{\cos[\alpha(1-T\eta^{-1})-\varphi]\sin\alpha\xi}{\sin\alpha}-\frac{\cos[\beta(1+T\eta^{-1})+\varphi]\sin\beta\xi}{\sin\beta}\right)$$
$$+\frac{P_o}{2\eta\tilde{\omega}}(\cos\left[\beta(\xi+T\eta^{-1})+\varphi\right]-\cos\left[\alpha(\xi-T\eta^{-1})-\varphi\right])H(\xi-T), \tag{53}$$

where $\alpha = \eta\tilde{\omega}/1-\eta$, $\beta = \eta\tilde{\omega}/1+\eta$.

In the particular case if η=1 instead of Eq. (51) one has

$$4\tilde{\omega}^2[w_I^A(\xi,T)]^{II}+\tilde{\omega}^4 w_I^A(\xi,T)$$
$$=-P_o\tilde{\omega}^2\sin(\tilde{\omega}T+\varphi)\delta(\xi-T)+2P_o\tilde{\omega}\cos(\tilde{\omega}T+\varphi)\delta^I(\xi-T). \tag{54}$$

After taking into account the boundary conditions (52), the solution to Eq. (54) takes the form

$$w_I^A(\xi,T) = \frac{-P_o}{4\tilde{\omega}\sin\frac{\tilde{\omega}}{2}}\left(\sin\left[\frac{\tilde{\omega}}{2}(\xi-1-T)-\varphi\right]+\sin\left[\frac{\tilde{\omega}}{2}(\xi+1+T)+\varphi\right]\right)$$
$$+\frac{P_o}{4\tilde{\omega}\sin\frac{\tilde{\omega}}{2}}\left(\sin\left[\frac{\tilde{\omega}}{2}(\xi+1+T)+\varphi\right]-\sin\left[\frac{\tilde{\omega}}{2}(\xi-1+T)+\varphi\right]\right)H(\xi-T). \tag{55}$$

Also the function $w_I^S(\xi,T)$ can be presented in a closed, analytical form. For the interval $i\eta \le T \le (i+1)\eta \le 1$ where $i=0,2,4,\ldots,n$, and $\eta<1$ ($v<v_s$) the function $w_I^S(\xi,T)$ is also a solution of the equation

$$(1-\eta^2)^2[w_I^S(\xi,T)]^{IV}+2\eta^2\tilde{\omega}^2(1+\eta^2)\left[w_I^S(\xi,T)\right]^{II}+\eta^4\tilde{\omega}^4 w_I^S(\xi,T)$$
$$=P_o\eta\sin\varphi\{(1-\eta^2)\delta^{II}[\xi-(T\eta^{-1}-i)]-\eta^2\tilde{\omega}^2\delta[\xi-(T\eta^{-1}-i)]\}-2P_o\eta^2\tilde{\omega}\cos\varphi\delta^I[\xi-(T\eta^{-1}-i)], \quad (56)$$

and for $i\eta \le T \le (i+1)\eta \le 1$, where $i=1,3,5,\ldots,n+1$, respectively, of the equation

$$(1-\eta^2)^2[w_I^S(\xi,T)]^{IV}+2\eta^2\tilde{\omega}^2(1+\eta^2)[w_I^S(\xi,T)]^{II}+\eta^4\tilde{\omega}^4 w_I^S(\xi,T)$$
$$=P_o\eta\sin\varphi\{(1-\eta^2)\delta^{II}[\xi-(i+1-T\eta^{-1})]-\eta^2\tilde{\omega}^2\delta[\xi-(i+1-T\eta^{-1})]\}-2P_o\eta^2\tilde{\omega}\cos\varphi\delta^I[\xi-(i+1-T\eta^{-1})]. \quad (57)$$

After solving Eq. (56) for $\eta\neq1$ one obtains for the boundary conditions (52)

$$w_I^S(\xi,T)=\frac{P_o}{2\eta\tilde{\omega}}\left(\frac{\cos[\beta(i+1-T\eta^{-1})-\varphi]\sin\beta\xi}{\sin\beta}-\frac{\cos[\alpha(i+1-T\eta^{-1})-\varphi]\sin\alpha\xi}{\sin\alpha}\right)$$
$$+\frac{P_o}{2\eta\tilde{\omega}}(\cos\left[\alpha(\xi+i-T\eta^{-1})-\varphi\right]-\cos\left[\beta(\xi+i-T\eta^{-1})-\varphi\right])\cdot H(\xi+1-T\eta^{-1}). \quad (58)$$

The solution to Eq. (57), for $\eta\neq1$, has the form

$$w_I^S(\xi,T)=\frac{P_o}{2\eta\tilde{\omega}}\left(\frac{\cos[\beta(i-T\eta^{-1})-\varphi]\sin\beta\xi}{\sin\beta}-\frac{\cos[\alpha(i-T\eta^{-1})-\varphi]\sin\alpha\xi}{\sin\alpha}\right)$$
$$+\frac{P_o}{2\eta\tilde{\omega}}\{\cos\left[\alpha(\xi-i-1+T\eta^{-1})+\varphi\right]-\cos\left[\beta(\xi-i-1+T\eta^{-1})+\varphi\right]\}H\left[\xi-(i+1-T\eta^{-1})\right]. \quad (59)$$

The function $w_{II}^A(\xi,T)$ can also be obtained in a closed form but it would have a compound form that depends on the values of the parameters; for this reason it has been presented in Appendix.

5. Numerical results

Figs. 3–10 present some results of vibration of the strings under moving, constant force. Two dimensionless parameters are assumed to be the same on all graphs, namely $P_0=1$ and $k_0=5$.On all figures the displacements of the top string (loaded by force) are presented by the solid line and the displacement of the bottom string is presented by the dashed line. Figs. 3–7 present the displacement of the strings if the velocity of the force is less than the velocity of the wave ($\eta<1$) for

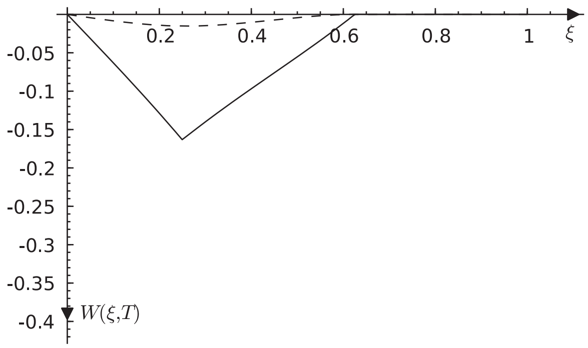

Fig.3. Displacements of the springs for $\eta=0.4$ and time $T=0.25$.

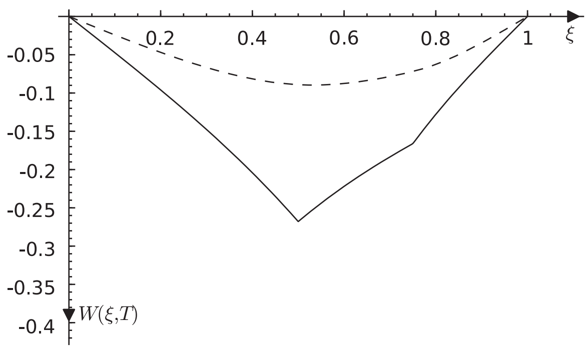

Fig.4. Displacements of the springs for $\eta=0.4$ and time $T=0.5$.

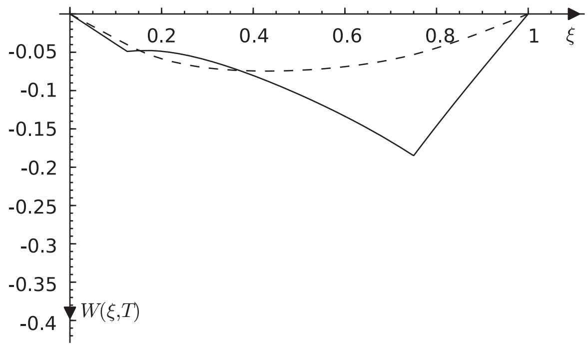

Fig.5. Displacements of the springs for η=0.4 and time T=0.75.

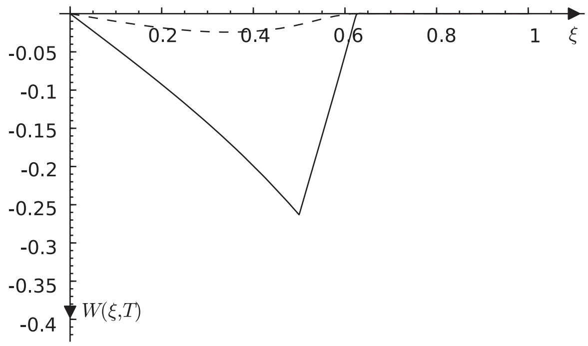

Fig.6. Displacements of the springs for η=0.8 and time T=0.5.

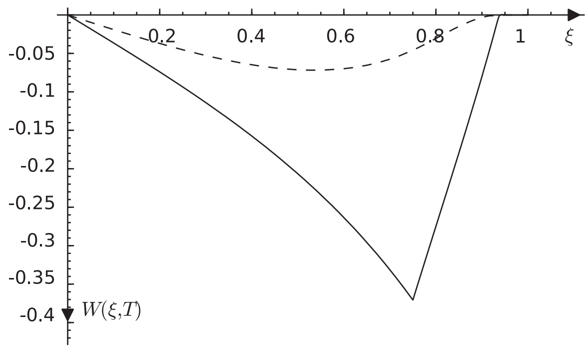

Fig.7. Displacements of the springs for η=0.8 and time T=0.75.

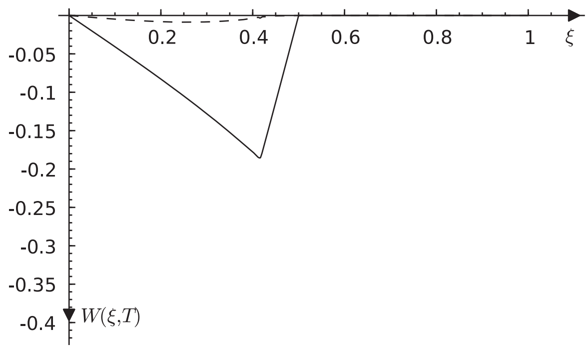

Fig.8. Displacements of the springs for η=1.2 and time T=0.5.

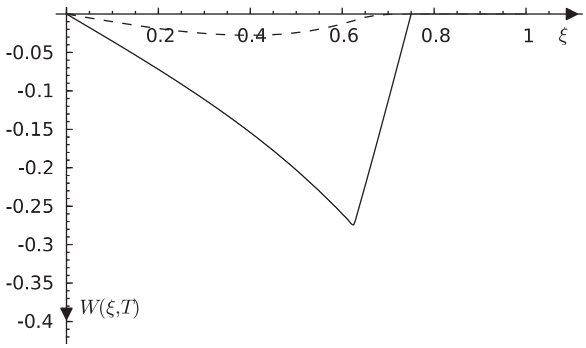

Fig.9. Displacements of the springs for η=1.2 and time T=0.75.

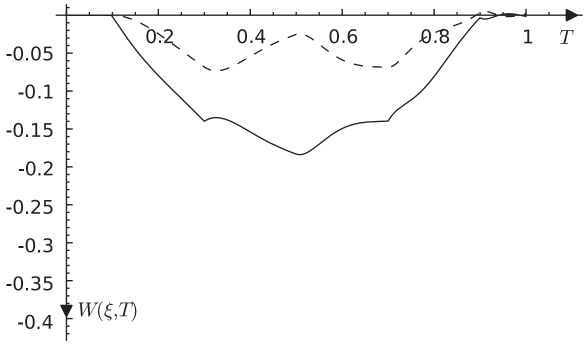

Fig. 10. Displacement of the springs for η=0.2 and ξ=0.5 as the function of time.

time T=0.25, T=0.5, and T=0,75. Figs. 8 and 9 present the displacement of the strings if the velocity of the force is bigger than the velocity of the wave ($\eta > 1$) for the time T=0.5 and T=0.75. Fig. 10 presents the displacement of the strings in the midpoint (ξ=0.5) as a function of time. The graphs for the loaded string (the top string) have two characteristic points, namely, in Figs. 3–9, the position of the moving force and the wave-front, in Fig. 10 the time when the force is directly at the midpoint and the time when the wave-front is directly at the midpoint.

6. Conclusions

The dynamics response of an elastically connected double-string complex system loaded by a constant or a harmonic oscillation force moving with a constant velocity has been studied. The classical solutions for transverse displacement of the strings have the form of sums of two infinite series. It has been shown that a part of the solution can be presented in a closed, analytical form. The closed, analytical solutions are derived from the fact that the solutions in the form of series are integrals not only of partial differential equations but also of some ordinary differential equations. The closed solutions take different forms depending if the velocity v of the moving force is smaller than, equal to or bigger than the wave velocity in the strings. This follows from the fact that in a string wave phenomena may occur. The presented closed solutions have an important meaning in the case when we consider the tension force in the string. The closed solutions allow one to analyze the vibrations phenomena due to moving forces without performing numerical calculations, see Fig. 2.

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Appendix

The function $w_{II}^{A}(\xi,T)$ can be also presented in a closed form after solving the equation

$$(1-\eta^2)^2[w_{II}^{A}(\xi,T)]^{IV}-2[(1-\eta^2)k_o-\eta^2\tilde{\omega}^2(1+\eta^2)][w_{II}^{A}(\xi,T)]^{II}+(k_o-\eta^2\tilde{\omega}^2)^2 w_{II}^{A}(\xi,T)$$
$$=-P_o\sin(\tilde{\omega}T+\varphi)[(1-\eta^2)\delta^{II}(\xi-T)+(\eta^2\tilde{\omega}^2-k_o)\delta(\xi-T)]+2P_o\eta^2\tilde{\omega}\cos(\tilde{\omega}T+\varphi)\delta^{I}(\xi-T), \quad (1A)$$

for $\eta \neq 1$ and taking into account the boundary conditions (52).

Let us introduce the constants and functions

$$b = 2\frac{(1-\eta^2)k_o - \eta^2\tilde{\omega}^2(1+\eta^2)}{(1-\eta^2)^2}, \quad g(T) = \frac{P_o \sin(\tilde{\omega}T+\varphi)}{(1-\eta^2)},$$
$$h(T) = \frac{P_o \sin(\tilde{\omega}T+\varphi)(\eta^2\tilde{\omega}^2 - k_o)}{(1-\eta^2)^2},$$
$$q(T) = \frac{2P_o\eta^2\tilde{\omega}\cos(\tilde{\omega}T+\varphi)}{(1-\eta^2)^2}, \quad c = \frac{(k_o-\eta^2\tilde{\omega}^2)^2}{(1-\eta^2)^2},$$
$$d_1^2 = \frac{b-\sqrt{b^2-4c}}{2}, \quad d_2^2 = \frac{b+\sqrt{b^2-4c}}{2}. \tag{2A}$$

The closed solution to Eq. (1A) for $d_1^2 > 0$ and $d_2^2 > 0$ has the form

$$\begin{aligned} w_{II}^A(\xi,T) = \frac{1}{d_1 d_2 (d_1^2 - d_2^2)} \{ & d_1 d_2 q(T)(\cosh[d_1(\xi-T)] - \cosh[d_2(\xi-T)])H(\xi-T) \\ & + \left(d_2\left(h(T)-d_1^2 g(T)\right)\sinh[d_1(\xi-T)] - d_1\left(h(T)-d_2^2 g(T)\right)\sinh[d_2(\xi-T)]\right)H(\xi-T) \\ & - \frac{d_1 d_2 q(T)\cosh[d_1(1-T)] + d_2\left(h(T)-d_1^2 g(T)\right)\sinh[d_1(1-T)]}{\sinh d_1}\sinh d_1\xi \\ & + \frac{d_1 d_2 q(T)\cosh[d_2(1-T)] + d_1\left(h(T)-d_2^2 g(T)\right)\sinh[d_2(1-T)]}{\sinh d_2}\sinh d_2\xi \Big\}, \end{aligned} \tag{3A}$$

for $d_1^2 < 0$ and $d_2^2 > 0$

$$\begin{aligned} w_{II}^A(\xi,T) = \frac{1}{d_1 d_2 (d_1^2 + d_2^2)} \{ & -d_1 d_2 q(T)(\cos[d_1(\xi-T)] - \cosh[d_2(\xi-T)])H(\xi-T) \\ & - \left(d_2\left(h(T)+d_1^2 g(T)\right)\sin[d_1(\xi-T)] - d_1\left(h-d_2^2 g\right)\sinh[d_2(\xi-T)]\right)H(\xi-T) \\ & + \frac{d_1 d_2 q(T)\cos[d_1(1-T)] + d_2\left(h(T)+d_1^2 g(T)\right)\sin[d_1(1-T)]}{\sin d_1}\sin d_1\xi \\ & - \frac{d_1 d_2 q(T)\cosh[d_2(1-T)] + d_1\left(h(T)-d_2^2 g(T)\right)\sinh[d_2(1-T)]}{\sinh d_2}\sinh d_2\xi \Big\} \end{aligned} \tag{4A}$$

and for $d_1^2 < 0$, $d_2^2 < 0$

$$\begin{aligned} w_{II}^A(\xi,T) = \frac{1}{d_1 d_2 (d_1^2 + d_2^2)} \{ & -d_1 d_2 q(T)(\cos[d_1(\xi-T)] - \cos[d_2(\xi-T)])H(\xi-T) \\ & - \left(d_2\left(h(T)+d_1^2 g(T)\right)\sin[d_1(\xi-T)] - d_1\left(h(T)+d_2^2 g(T)\right)\sin[d_2(\xi-T)]\right)H(\xi-T) \\ & + \frac{d_1 d_2 q(T)\cos[d_1(1-T)] + d_2\left(h(T)+d_1^2 g(T)\right)\sin[d_1(1-T)]}{\sin d_1}\sin d_1\xi \\ & - \frac{d_1 d_2 q(T)\cos[d_2(1-T)] + d_1\left(h(T)+d_2^2 g(T)\right)\sin[d_2(1-T)]}{\sin d_2}\sin d_2\xi \Big\}. \end{aligned} \tag{5A}$$

In the particular case if η=1, Eq. (1A) is reduced to the form

$$\begin{aligned} & 4\tilde{\omega}^2[w_{II}^A(\xi,T)]^{II} + (k_o - \tilde{\omega}^2)^2 w_{II}^A(\xi,T) \\ & = -P_o \sin(\tilde{\omega}T+\varphi)(\tilde{\omega}^2 - k_o)\delta(\xi-T) + 2P_o\tilde{\omega}\cos(\tilde{\omega}T+\varphi)\delta^I(\xi-T), \end{aligned} \tag{6A}$$

and the closed solution has the form

$$\begin{aligned} w_{II}^A(\xi,T) = & \frac{-P_o}{4\tilde{\omega}\sin a}\left\{\sin\left[a(\xi+1-T) - \tilde{\omega}T - \varphi\right] + \sin\left[a(\xi-1+T) + \tilde{\omega}T + \varphi\right]\right\} \\ & + \frac{P_o}{4\tilde{\omega}\sin a}\left\{\sin\left[a(\xi+1-T) - \tilde{\omega}T - \varphi\right] - \sin\left[a(\xi-1-T) - \tilde{\omega}T - \varphi\right]\right\}H(\xi-T), \end{aligned} \tag{7A}$$

where

$$a = \frac{k_o - \tilde{\omega}^2}{2\tilde{\omega}}. \tag{8A}$$

For the general integral $w_{II}^S(\xi,T)$, similarly as in the case of a constant moving force, one cannot find a closed solution by the method presented above.

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